Student Number

Hunters Hill
High School

2023 **Trial Examination**

Mathematics Extension 1

General Instructions:	Reading time – 10 minutes Working time – 2 hours Write using black pen NESA approved calculators may be used A reference sheet is provided For questions in Section II, show relevant mathematical reasoning and/or calculations
Total Marks: 70	**Section I – 10 marks (pages 3 – 8)** Attempt all Questions 1 – 10 Allow about 15 minutes for this section **Section II – 60 marks (pages 9 – 15)** Attempt all Questions 11 – 14 Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Question 1 – 10.

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1. Given $\tan\theta = \dfrac{1}{3}$, what is the exact value of $\tan\left(\theta + \dfrac{\pi}{3}\right)$?

(A) $\dfrac{\sqrt{3}+3}{3\sqrt{3}-1}$

(B) $\dfrac{\sqrt{3}-3}{3\sqrt{3}+1}$

(C) $\dfrac{1+3\sqrt{3}}{3-\sqrt{3}}$

(D) $\dfrac{1-3\sqrt{3}}{3+\sqrt{3}}$

2. Which of the following expressions is equivalent to $\displaystyle\int \frac{-2}{\sqrt{1-4x^2}}\,dx$?

(A) $\dfrac{1}{2}\sin^{-1}(2x) + c$

(B) $\sin^{-1}(2x) + c$

(C) $\dfrac{1}{2}\cos^{-1}(2x) + c$

(D) $\cos^{-1}(2x) + c$

3. Jack starts at the origin and walks along vector $2\underset{\sim}{i} + 3\underset{\sim}{j}$ and then turns and walks along $4\underset{\sim}{i} - 2\underset{\sim}{j}$. How far is Jack from the origin?

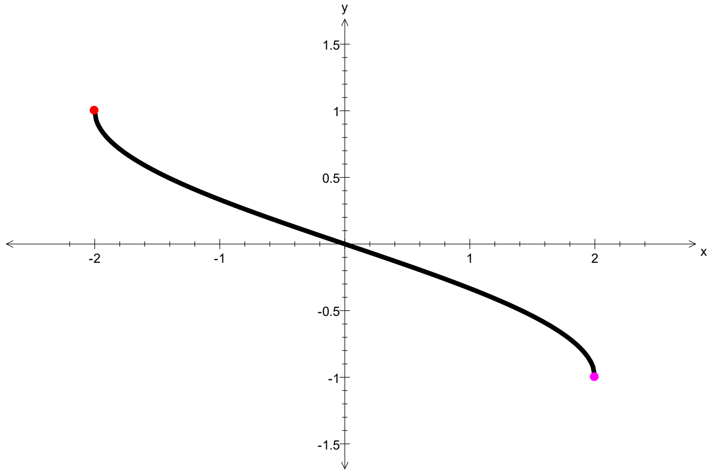
(A) 5

(B) $\sqrt{11}$

(C) $\sqrt{37}$

(D) $\sqrt{61}$

4. The function shown in the diagram below has equation $y = A\sin^{-1} Bx$.
Which of the following is true?



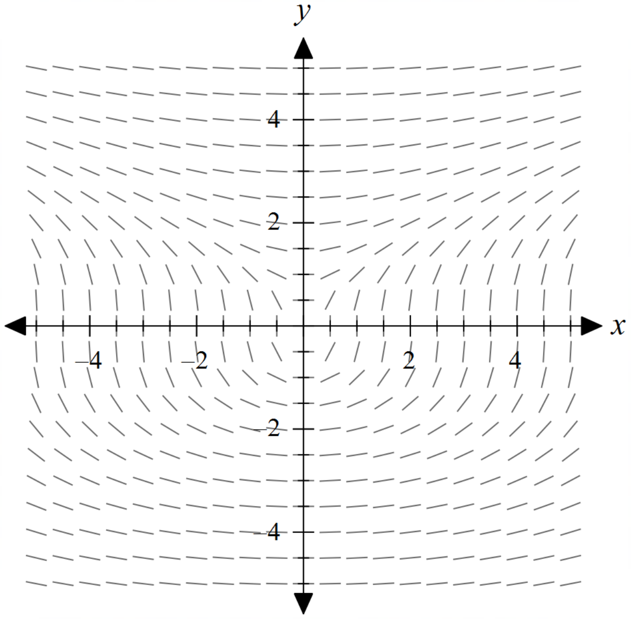
(A) $A = 1, B = \dfrac{1}{2}$

(B) $A = -1\,, B = 2$

(C) $A = \dfrac{-2}{\pi}\,, B = \dfrac{1}{2}$

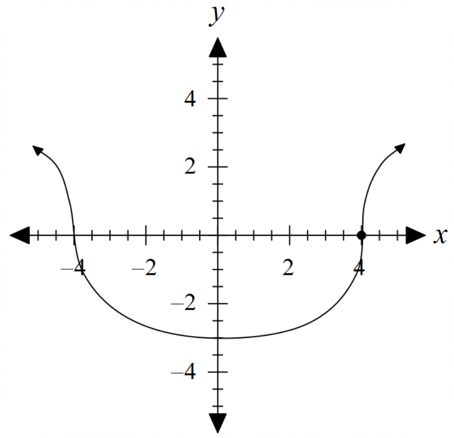
(D) $A = \dfrac{2}{\pi}\,, B = 2$

5. The graph of the direction field of a differential is shown below.

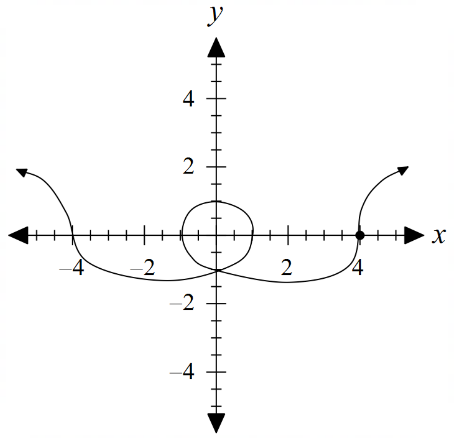


Which of the following best represents the particular solution that passes through the point $(4, 0)$?

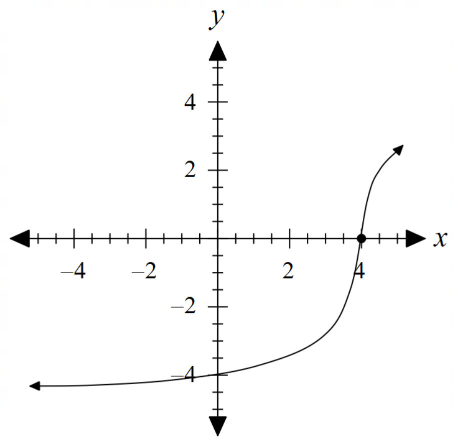
(A)



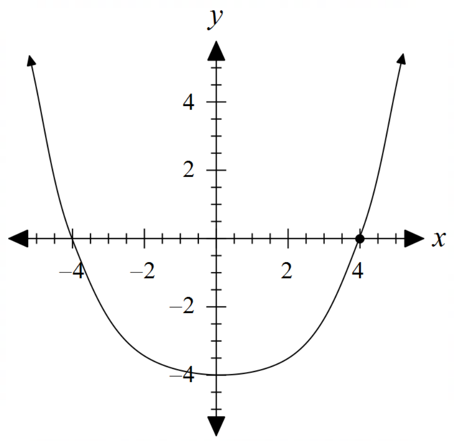
(B)



(C)



(D)



6. A school committee consists of 8 members and a chairperson. The members are selected from 12 students. The chairperson is selected from 4 teachers. In how many ways could the committee be selected?

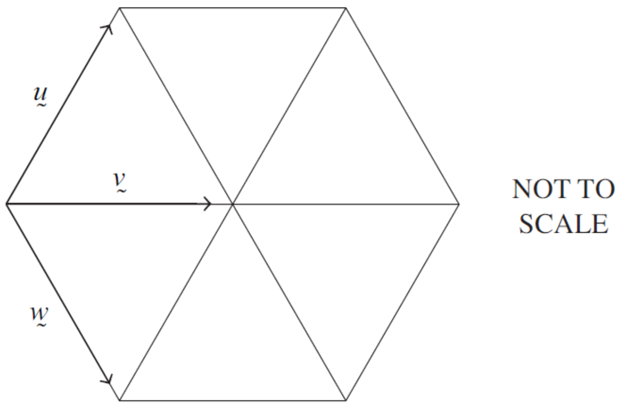
(A) ${}^{12}C_8 + {}^{4}C_1$

(B) ${}^{12}P_8 + {}^{4}P_1$

(C) ${}^{12}P_8 \times {}^{4}P_1$

(D) ${}^{12}C_8 \times {}^{4}C_1$

7. Six equilateral triangles form a hexagon with side lengths of 4 cm. The vectors $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$ are shown in the diagram.



Which of the following is the value of $\underset{\sim}{u} \cdot \left(\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}\right)$?

(A) 8

(B) 16

(C) 32

(D) 48

8. The expression $2\cos x - 3\sin x$ is written in the form $R\cos(x+\theta)$, where $R > 0$ and $0 \le \theta \le \frac{\pi}{2}$. What is the value of $\tan\theta$?

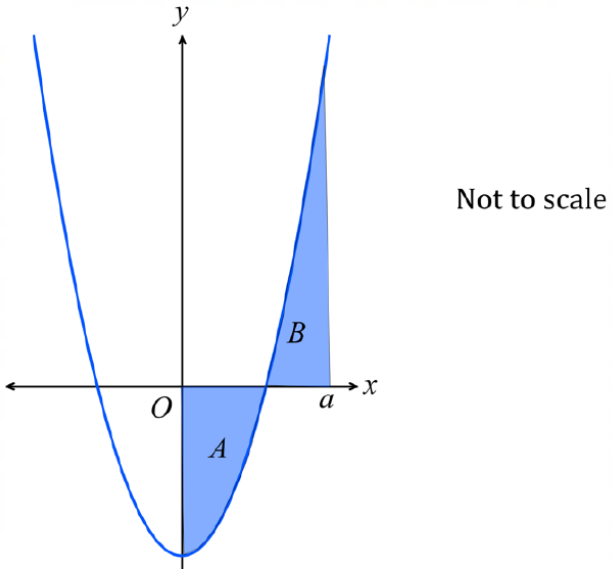
(A) $-\dfrac{3}{2}$

(B) $-\dfrac{2}{3}$

(C) $\dfrac{2}{3}$

(D) $\dfrac{3}{2}$

9. The graph of $y = x^2 - 4$ is shown below. The area of the region A is equal to the area of the region B. What is the value of a?



(A) 6

(B) $\sqrt{6}$

(C) $2\sqrt{3}$

(D) 12

10. Mathematical induction is used to prove that

$$\frac{2}{1 \times 2} + \frac{2}{2 \times 3} + \frac{2}{3 \times 4} + \cdots + \frac{2}{n \times (n+1)} = \frac{2n}{n+1}$$

for all positive integers n.

Which of the following is the correct expression for part of the induction proof?

(A) $LHS = \dfrac{2k}{k+1} + \dfrac{2}{k+1}$

(B) $LHS = \dfrac{2k}{k+1} + \dfrac{2}{k+2}$

(C) $LHS = \dfrac{2k}{k+1} + \dfrac{2}{(k+1)(k+2)}$

(D) $LHS = \dfrac{2k}{k+1} + \dfrac{2}{(k+1)(k+3)}$

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Question 11 (16 marks) Begin a new Writing Booklet.

a. Find the value of the constant a such that $8750x^4$ is a term in the binomial expansion of $(a + 2x)^7$. **3**

b. **(i)** Use the substitution $x = \tan^2\theta$ to show that $\displaystyle\int_0^1 \frac{\sqrt{x}}{(1+x)^2}\,dx = \int_0^{\frac{\pi}{4}} 2\sin^2\theta\, d\theta$. **2**

(ii) Hence find in simplest exact form the value of $\displaystyle\int_0^1 \frac{\sqrt{x}}{(1+x)^2}\,dx$. **2**

c. Solve $(x^2 - 1)(x + 3) < 0$. **3**

d. Find in the form $y = f(x)$ the solution of the differential equation $\dfrac{dy}{dx} = \dfrac{2e^{-\frac{1}{2}y}}{\cos^2 x}$ given that $y = \ln 3$ when $x = \frac{\pi}{3}$. **3**

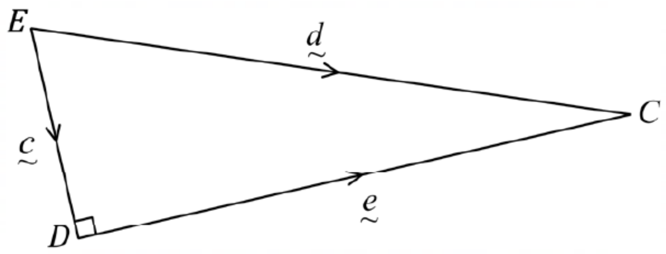
e. In a particular country town, the proportion of employment in the farming industry is 0.62. Five people aged 15 years and older are selected at random from the town. Calculate the probability that the proportion of workers in the farming industry in the sample is greater than 0.5, correct your answer to 2 decimal places. **3**

End of Question 11

Question 12 (13 marks) Begin a new Writing Booklet.

a. Use Mathematical Induction to prove that $13^n + 6^{n-1}$ is divisible by 7 for all positive integers $n \geq 1$. **3**

b. ΔDEC has a right angle at D as shown below.



Show that:

i. $\left|\underset{\sim}{d}\right|^2 = \underset{\sim}{e} \cdot \underset{\sim}{e} + 2\left(\underset{\sim}{e} \cdot \underset{\sim}{c}\right) + \underset{\sim}{c} \cdot \underset{\sim}{c}$ **2**

ii. $\left|\underset{\sim}{d}\right|^2 = \left|\underset{\sim}{e}\right|^2 + \left|\underset{\sim}{c}\right|^2$ **1**

c. A particular instant noodle company states that, at most, 3% of all their noodle packets marked 85 grams may weigh less than 82 grams. A random sample of 250 noodle packets is selected and weighted.

i. Find the mean and standard deviation for this distribution of sample proportions. Give the mean correct to two decimal places and the standard deviation correct to four decimal places. **2**

ii. Estimate the probability that, at most, 2% of the noodle packets weigh less than 82 grams using the z-score table below. **2**

Part of a table of P $(Z < z)$ values, where Z is a standard normal variable, is shown.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8461	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621

Question 12 continues onto next page

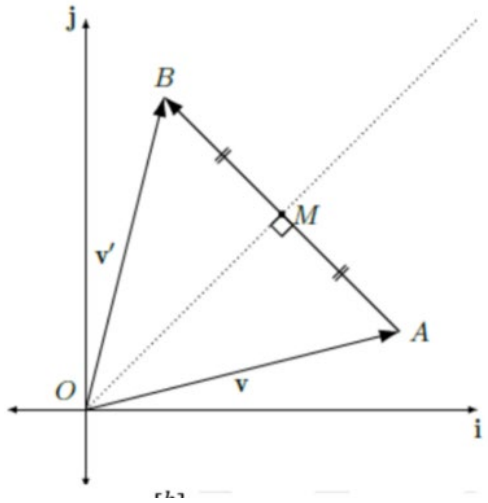
d. A curve has a gradient function $\dfrac{4e^{2x}}{1+e^{4x}}$ and passes through the point $\left(0\,,\frac{\pi}{2}\right)$. Use the substitution $u = e^{2x}$ to find its equation. **3**

End of Question 12

Question 13 (16 marks) Begin a new Writing Booklet.

a. The diagram shows the vector $v = \binom{a}{b}$ · The vector v' is its reflection about the dotted line, $y = x$. M bisects AB and lies on $y = x$.

Using projections, show that $v' = \binom{b}{a}$. **3**



b. The number of mobile phones, N owned in a certain community after t years, may be modelled by $\log_e N = 6 - 3e^{-0.4t}$, $t > 0$.

i. Show that $\log_e N = 6 - 3e^{-0.4t}$ is a solution to the differential equation **2**

$$\frac{1}{N}\frac{dN}{dt} + 0.4\log_e N - 2.4 = 0.$$

ii. The differential equation in **i** above can also be written in the form of **2**

$$\frac{dN}{dt} = 0.4N(6 - \log_e N).$$

By using the chain and product rule, show that $\dfrac{d^2N}{dt^2} = \dfrac{4N}{25}(6 - \log_e N)(5 - \log_e N)$.

iii. The graph of N as a function of t has a point of inflection. Find the coordinates of this point. Give the value of t to one decimal place and the value of N to the nearest integer. **2**

iv. By finding the initial and the limiting number of mobile phones that would eventually be owned in the community, sketch the graph of N as a function of t on a Cartesian plane for $0 \le t \le 15$. Showing all key features. **3**

Question 13 continues onto next page

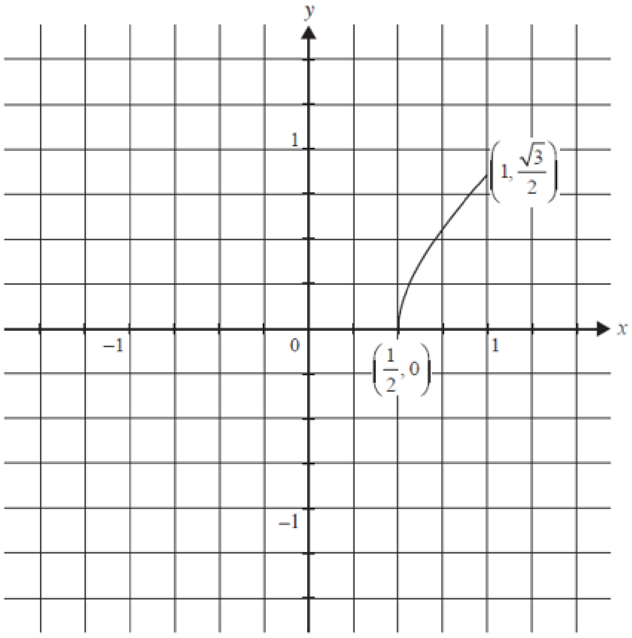
c. **i.** Show that $\dfrac{\sin 2\theta \cos\theta - \sin\theta \cos 2\theta}{\sin 6\theta \cos 2\theta - \sin 2\theta \cos 6\theta} = \dfrac{1}{4}\sec\theta \sec 2\theta$. **3**

ii. If $\dfrac{1}{4}\sec\theta \sec 2\theta = -\dfrac{1}{4}$, find $\theta, 0 \leq \theta \leq 2\pi$. **3**

End of Question 13

Question 14 (15 marks) Begin a new Writing Booklet.

a. The water in the fountain is modelled by part of a graph $y = \frac{1}{2}\sqrt{4x^2 - 1}$ as shown below, where y represents the depth of water.



i. Show that the volume, V m^3, of water in the fountain when it is filled to a depth of h metres is given by **3**

$$V = \frac{\pi}{4}\left(\frac{4}{3}h^3 + h\right)$$

ii. The fountain is initially empty. A vertical jet of water in the centre fills the fountain at a rate of 0.04 m^3/s and, at the same time, water flows out from the bottom of the fountain at a rate of $0.05\sqrt{h}$ m^3/s when the depth is h metres. Show that **3**

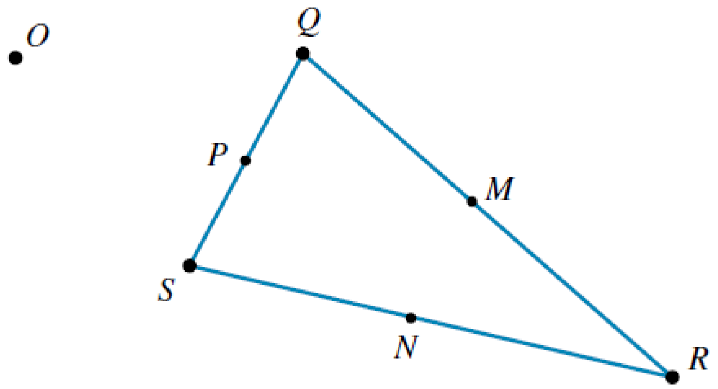
$$\frac{dh}{dt} = \frac{4 - 5\sqrt{h}}{25\pi(4h^2 + 1)}$$

iii. Express the time taken for the depth of the above fountain to reach 0.25m as a definite integral. **Do not** evaluate. **1**

iv. How far from the top of the fountain does the water level ultimately stabilise? Give your answer in metres, correct to two decimal places. **2**

Question 14 continues onto next page

b. The diagram shows a triangle with vertices Q, R and S. O is the origin, and the vector $\overrightarrow{OQ} = \underset{\sim}{q}$ and $\overrightarrow{OS} = \underset{\sim}{s}$. Given that M is the midpoint of the $\overrightarrow{QR}$, N is the midpoint of the line segment RS and P is the midpoint of the line segment QS, prove that the quadrilateral $MNPQ$ is a parallelogram. **3**



c. Bags of lollipops are supposed to contain 50 lollipops each. Production records indicate that 92% of bags contain 50 lollipops. A batch of 15 bags is sampled. If more than 2 bags do not contain exactly 50 lollipops, production is stopped.

i. Find the probability, correct to three decimal places, that exactly 2 of the 15 bags selected do not contain 50 lollipops. **1**

ii. Find the probability, correct to three decimal places, the production is stopped. **2**

End of Examination

2023 Maths Ext 1 Trial Soln

Monday, 31 July 2023 11:34 AM

Section 1

Q1 C

$\tan(\theta + \frac{\pi}{3})$

$= \frac{\tan\theta + \tan\frac{\pi}{3}}{1 - \tan\theta\tan\frac{\pi}{3}}$

$= \frac{1 + 3\sqrt{3}}{3 - \sqrt{3}}$

Q2 D

$\int \frac{-2}{\sqrt{1-(2x)^2}}\,dx$ ref sheet.

$= \cos^{-1}(2x) + C$

Q3. C

$2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{i} - 2\underset{\sim}{j}$

$= 6\underset{\sim}{i} + \underset{\sim}{j}$

mag : $\sqrt{6^2+1^2} = \sqrt{37}$

Q4. C

$\frac{y}{A} = \sin^{-1} Bx$

$\frac{\pi}{2} \times A = -1$

$\therefore A = \frac{-2}{\pi}$

$\frac{1}{B} = 2$

$\therefore B = \frac{1}{2}$

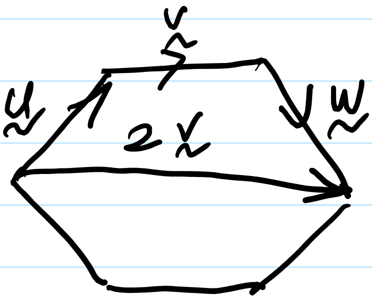
Q5

A

Q6 D

selection from the group and the order is not important

Q7.



B

$\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w} = 2\underset{\sim}{v}$

$\underset{\sim}{u} \cdot 2\underset{\sim}{v} \Rightarrow \cos\theta = \dfrac{\underset{\sim}{u} \cdot 2\underset{\sim}{v}}{|\underset{\sim}{u}||2\underset{\sim}{v}|}$

$\therefore \underset{\sim}{u} \cdot 2\underset{\sim}{v} = |\underset{\sim}{u}||2\underset{\sim}{v}| \cos\theta$

all angles are equilateral $\therefore \theta = \frac{\pi}{3}$

$\therefore |\underset{\sim}{u}| = 4, \ |2\underset{\sim}{v}| = 8,$

$\underset{\sim}{u} \cdot 2\underset{\sim}{v} = 4 \times 8 \times \cos\frac{\pi}{3}$

$= 16$

Q8. D $2\cos x - 3\sin x = R\cos(x + \theta)$

$2\cos x = R\cos x \cos\theta$

$3\sin x = R\sin x \sin\theta$

$R\cos\theta = 2 \quad \therefore \tan\theta = \frac{3}{2}$

$R\sin\theta = 3$

Monday, 31 July 2023 11:35 AM

Q9. C

$A = \int_a^b y\,dx = \int_0^a (x^2-4)\,dx$

$0 = \left[\frac{x^3}{3} - 4x\right]_0^a$

$\frac{a^3}{3} = 4a$

$a^2 = 12 \quad a>0$

$a = 2\sqrt{3}$

Q10. C

$\boxed{\text{Section II}}$

Q11 a) ${}^7C_4(2x)^4a^3 = 560a^3x^4$ $\boxed{\text{1 mark}}$

$a^3 = \frac{8750}{650} = \frac{125}{8}$ $\boxed{\text{1 mark}}$

$a = \frac{5}{2}$ $\boxed{\text{1 mark}}$

b) i) $x = \tan^2\theta$ $\quad x=0 \Rightarrow \theta = 0$ $\boxed{\text{1 mark}}$

$\frac{dx}{d\theta} = 2\tan\theta\sec^2\theta$ $\quad x=1 \Rightarrow \theta = \frac{\pi}{4}$

$$\int_0^1 \frac{\sqrt{x}}{(1+x)^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{\tan\theta}{(1+\tan^2\theta)^2}\, 2\tan\theta\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan\theta}{(\sec^2\theta)^2}\, 2\tan\theta\,\underline{\sec^2\theta\,d\theta}$$ $\boxed{\text{1 mark}}$

$$= \int_0^{\frac{\pi}{4}} 2\sin^2\theta\,d\theta$$

Q11 b ii) $\int_0^1 \frac{\sqrt{x}}{(1+x^2)^2}\,dx$

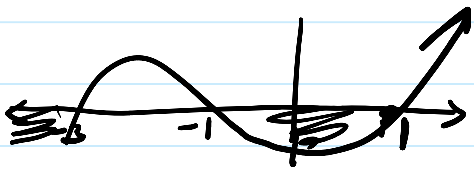
$= \int_0^{\frac{\pi}{4}} 2\sin^2\theta \, d\theta$

$= \int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\, d\theta$ 1 mark

$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - \frac{1}{2}$ 1 mark

Q11 c. $(x^2-1)(x+3) < 0$



1 mark

$x^2-1=0 \qquad x+3=0$

$x = \pm 1 \qquad x = -3$

$\therefore \quad x < -3 \text{ or } -1 < x < 1$

1 mark for each correct region

Q11 d. $\frac{dy}{dx} = \frac{2e^{-\frac{1}{2}y}}{\cos^2 x}$

$\frac{dy}{dx} = \sec^2 x \, 2e^{-\frac{1}{2}y}$

$\frac{dy}{2e^{-\frac{1}{2}y}} = \sec^2 x \, dx$ 1 mark

$\int \frac{1}{2} e^{\frac{1}{2}y}\, dy = \int \sec^2 x \, dx$

$e^{\frac{1}{2}y} = \tan x + C$

when $y = \ln 3$, $x = \frac{\pi}{3}$,

$e^{\frac{1}{2}\ln 3} = \sqrt{3} + C$

$C = 0$

$\therefore e^{\frac{1}{2}y} = \tan x + 0$ 1 mark

$\frac{1}{2}y = \ln(\tan x)$

$y = 2\ln(\tan x)$

Q11 e. $P(\hat{p} > 0.5) = ?$ $p = 0.62$ $q = 1 - 0.62 = 0.38$
$n = 5$, $np = 5 \times 0.62 = 3.1 < 5$, no normal distr

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$
$P(\hat{p})$				✓	✓	✓

1 mark

$P(\hat{p} > 0.5) = P(\hat{p} = \frac{3}{5}) + P(\hat{p} = \frac{4}{5}) + P(\hat{p} = \frac{5}{5})$

1 mark

$= {}^5C_3\, 0.62^3\, 0.38^2 + {}^5C_4\, 0.62^4\, 0.38^1 + {}^5C_5\, 0.62^5\, 0.38^0$

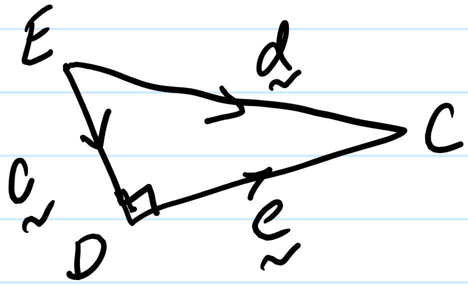
$P(\hat{p} > 0.5) = 0.717$ 1 mark

Q12 a. let the proposition be P_n such that $13^n + 6^{n-1} \div 7M$ for all positive integer M.

for P_1: $13^1 + 6^0 = 14 = 7 \times 2$ 1 mark
$\therefore P_1$ is true.
If $P(k)$ is true: $13^k + 6^{k-1} = 7M$ for integer $M > 0$ *.
RTP
$P(k+1) = 13^{k+1} + 6^{k+1-1}$
$= 13 \times 13^k + 6^k$
$= 13 \times 13^k + 6 \times 6^{k-1}$ 1 mark
$= (7+6) \times 13^k + 6 \times 6^{k-1}$
$= 7 \times 13^k + 6(13^k + 6^{k-1})$
1 mark $= 7 \times 13^k + 6 \times 7M$ use P_k *
$= 7(13^k + 6M)$ where $13^k + 6M$ is integral
$\therefore P(k) \Rightarrow P(k+1)$
$\therefore P(n)$ is true for all $n \geq 1$ by induction.

Monday, 31 July 2023 11:35 AM

Q12 b.



i) $|\underset{\sim}{d}|^2$

$= \underset{\sim}{d} \cdot \underset{\sim}{d}$

$= (\underset{\sim}{c} + \underset{\sim}{e}) \cdot (\underset{\sim}{c} + \underset{\sim}{e})$ 1 mark

$= \underset{\sim}{c} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{e} + \underset{\sim}{e} \cdot \underset{\sim}{c} + \underset{\sim}{e} \cdot \underset{\sim}{e}$

$|\underset{\sim}{d}|^2 = \underset{\sim}{e} \cdot \underset{\sim}{e} + 2(\underset{\sim}{c} \cdot \underset{\sim}{e}) + \underset{\sim}{c} \cdot \underset{\sim}{c}$

1 mark

ii) from (i) $\underset{\sim}{e} \cdot \underset{\sim}{c} = 0$ (ie $\cos 90° = 0$)

1 mark

$|\underset{\sim}{d}|^2 = |\underset{\sim}{e}|^2 + 2 \times 0 + |\underset{\sim}{c}|^2$

$\therefore |\underset{\sim}{d}|^2 = |\underset{\sim}{e}|^2 + |\underset{\sim}{c}|^2$

c. i) $p = 3\% = 0.03$ $\hat{p}$ = percentage of noodle weighs less than 82g

$n = 250$

$np = 0.03 \times 250 = 7.5 > 5$

$nq = 0.97 \times 250 = 242.5 > 5$

$E(\hat{p}) = p = 0.03$ 1 mark

$\sigma(\hat{p}) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{0.03 \times 0.97}{250}}$

$\sigma(\hat{p}) = 0.0108$ 1 mark

c) ii.

$P(\hat{p} \leq 2\%)$

$z = \dfrac{0.02 - 0.03}{0.0108}$ 1 mark

$z \simeq -0.9269$

$\therefore$ use the table.

$P(z \leq -0.93) = 1 - P(z \leq 0.93)$

$= 1 - 0.8238$ 1 mark

$= 0.1762$

Therefore, the probability that at most 2% weigh less than 82 g is 0.1762.

d) $\dfrac{dy}{dx} = \dfrac{4e^{2x}}{1+e^{4x}}$

$y = \displaystyle\int \frac{4e^{2x}}{1+e^{4x}}\, dx$

$= 2\displaystyle\int \frac{2e^{2x}}{1^2+(e^{2x})^2}\, dx$ 1 mark

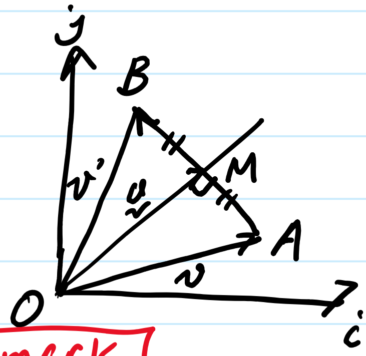
$y = 2\tan^{-1}(e^{2x}) + c$ 1 mark

when $x = 0$, $y = \frac{\pi}{2} = 2\tan^{-1}(1) + c$

$\therefore c = 0$

$\therefore y = 2\tan^{-1}(e^{2x})$ 1 mark

Q13 a. let $\hat{\underset{\sim}{u}}$ be the unit vector in the direction of $\overrightarrow{OM}$ $\underset{\sim}{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\underset{\sim}{u} = \underset{\sim}{\hat{i}} + \underset{\sim}{\hat{j}}$



$\overrightarrow{MA}$ is $\perp$ $\text{proj}_{\underset{\sim}{u}} \overrightarrow{OA}$

$$\overrightarrow{MA} = \overrightarrow{OA} - \text{proj}_{\underset{\sim}{u}} \overrightarrow{OA}$$

$$= \underset{\sim}{v} - \frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{|\underset{\sim}{u}|^2} \hat{\underset{\sim}{u}}$$ 1 mark

$$= \begin{pmatrix} a \\ b \end{pmatrix} - \frac{\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{(\sqrt{1^2+1^2})^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} - \frac{a+b}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ 1 mark

$$\underset{\sim}{v}' = \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$= \overrightarrow{OA} - 2\overrightarrow{MA}$$ 1 mark

$$= \begin{pmatrix} a \\ b \end{pmatrix} - 2\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a+b \\ a+b \end{pmatrix}$$

$$= \begin{pmatrix} a - 2a + a + b \\ b - 2b + a + b \end{pmatrix}$$

$$= \begin{pmatrix} b \\ a \end{pmatrix} \quad \therefore \underset{\sim}{v}' = \begin{pmatrix} b \\ a \end{pmatrix}$$

b. i) sub. $\ln N = 6 - 3e^{-0.4t}$ into the DE

$$\ln N = 6 - 3e^{-0.4t}$$

$$N = e^{6-3e^{-0.4t}}$$

$$\frac{dN}{dt} = 1.2e^{-0.4t} e^{6-3e^{-0.4t}}$$

$$\frac{1}{N}\frac{dN}{dt} = \frac{1}{\cancel{e^{6-3e^{-0.4t}}}} \times 1.2e^{-0.4t} \cancel{e^{6-3e^{-0.4t}}}$$ 1 mark

$$= 1.2e^{-0.4t}$$

b i) cont'

$$\frac{1}{N}\frac{dN}{dt} + 0.4\ln N - 2.4 = 0$$ 1 mark

$$LHS = 1.2e^{-0.4t} + 0.4(6-3e^{-0.4t}) - 2.4$$
$$= 2.4 - 2.4$$
$$= 0 = RHS.$$

ii)
$$\frac{d^2N}{dt^2} = \frac{d}{dt}\left(\frac{dN}{dt}\right)$$
$$= \frac{d}{dt}\left(0.4N(6-\ln N)\right)$$ 1 mark
$$= 0.4\frac{dN}{dt}(6-\ln N) + 0.4N\frac{d}{dt}(6-\ln N)$$
$$= 0.4\frac{dN}{dt}(6-\ln N) + 0.4N\times\frac{-1}{N}\frac{dN}{dt}$$
$$= \frac{dN}{dt}\left(0.4(6-\ln N) - 0.4\right)$$
$$= \frac{dN}{dt}(2-0.4\ln N)$$

OR
$$\frac{d^2N}{dt^2} = \frac{d\left(\frac{dN}{dt}\right)}{dN}\times\frac{dN}{dt}$$
$$\frac{d\left(\frac{dN}{dt}\right)}{dt} = 0.4(6-\ln N) - 0.4N\times\left(-\frac{1}{N}\right)$$
$$\frac{d\left(\frac{dN}{dt}\right)}{dN} = 2 - 0.4\ln N$$

Sub in $\frac{dN}{dt} = 0.4N(6-\ln N)$ 1 mark

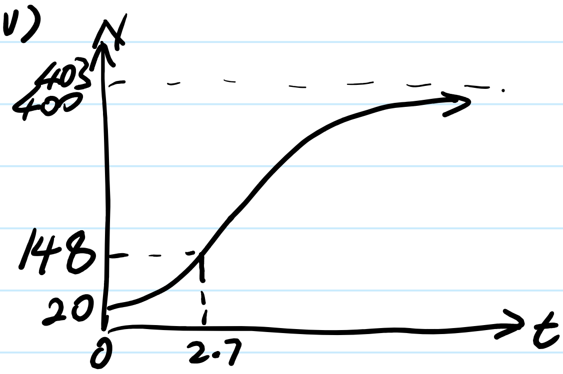
$$\frac{d^2N}{dt^2} = 0.4N(6-\ln N)(2-0.4\ln N)$$
$$= 0.4N\times 0.4(6-\ln N)(5-\ln N)$$
$$\frac{d^2N}{dt^2} = \frac{4N}{25}(6-\ln N)(5-\ln N)$$

iii) for point of inflection
$$\frac{d^2N}{dt^2} = 0$$

since $\ln N \neq 6$ 1 mark
$$\ln N = 5 \Rightarrow N = e^5 \approx 148$$
$$5 = 6 - 3e^{-0.4t} \Rightarrow e^{-0.4t} = \frac{1}{3}$$ $t = \frac{1}{0.4}\ln 3$
$t \simeq 2.7$ yrs
1 mark

Monday, 31 July 2023 11:35 AM

iv)



$\boxed{\text{1 mark correct shape}}$

at $t=0$, $\ln N = 6-3e^{0} = 3$ $\boxed{\text{1 mark}}$

$N = e^3 \simeq 20$

as $t \to \infty$, $\ln N \to 6-3e^{-\infty} = 6$ $\boxed{\text{1 mark}}$

$N = e^6 \simeq 403$

point of inflection $(2.7, 148)$

C. i) $LHS = \dfrac{\sin 2\theta \cos\theta - \sin\theta \cos 2\theta}{\sin 6\theta \cos 2\theta - \sin 2\theta \cos 6\theta}$

$= \dfrac{\sin(2\theta - \theta)}{\sin(6\theta - 2\theta)}$ $\boxed{\text{1 mark}}$

$= \dfrac{\sin\theta}{\sin 4\theta}$

$= \dfrac{\sin\theta}{\sin(2\times 2\theta)}$ $\boxed{\text{1 mark (double angle)}}$

$= \dfrac{\sin\theta}{2\sin 2\theta \cos 2\theta}$

$= \dfrac{\cancel{\sin\theta}}{4\cancel{\sin\theta}\cos\theta}\sec 2\theta$ $\boxed{\text{1 mark}}$

$= \frac{1}{4}\sec\theta\sec 2\theta = RHS$

ii) $\frac{1}{4}\sec\theta \sec 2\theta = -\frac{1}{4}$

$\dfrac{1}{\cos\theta \cos 2\theta} = -1$

$\cos\theta(2\cos^2\theta - 1) = -1$ $\boxed{\text{1 mark}}$

$2\cos^3\theta - \cos\theta + 1 = 0$

$(\cos\theta + 1)(2\cos^2\theta - 2\cos\theta + 1) = 0$ $\boxed{\text{1 mark}}$

$\therefore \cos\theta + 1 = 0 \Rightarrow \theta = \frac{3\pi}{2}$

1 mark

$2\cos^2\theta - 2\cos\theta + 1 = 0 \Rightarrow$ no soln.

Q14 a.i) $y = \frac{1}{2}\sqrt{4x^2-1}$

$4y^2 = 4x^2 - 1 \Rightarrow x^2 = \frac{1}{4}(1+4y^2)$

$V = \frac{\pi}{4}\int_0^h (1+4y^2)\,dy$

$V = \frac{\pi}{4}\left[y + \frac{4y^3}{3}\right]_0^h$

$V = \frac{\pi}{4}\left(h + \frac{4h^3}{3}\right)$

ii) inflow rate: $0.04\,m^3/s$

outflow rate: $0.05\sqrt{h}\ m^3/s$

$\frac{dV}{dt}$ = inflow − outflow.

$= 0.04 - 0.05\sqrt{h}$

$= \frac{1}{100}(4 - 5\sqrt{h})$ 1 mark

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$V = \frac{\pi}{4}\left(h + \frac{4h^3}{3}\right)$

$\frac{dV}{dh} = \frac{\pi}{4}(1 + 4h^2)$ 1 mark

$\therefore \frac{dh}{dt} = \frac{1}{\frac{\pi}{4}(1+4h^2)} \times \frac{1}{100}(4 - 5\sqrt{h})$

$\frac{dh}{dt} = \frac{4 - 5\sqrt{h}}{25\pi(4h^2+1)}$ 1 mark

iii) $\frac{dt}{dh} = \frac{25\pi(4h^2+1)}{4 - 5\sqrt{h}}$

$t = \int_0^{0.25} \frac{25\pi(4h^2+1)}{4 - 5\sqrt{h}}\,dh$

1 mark

iv) The water level stabilise when

$4-5\sqrt{h}=0$

$5\sqrt{h}=4$

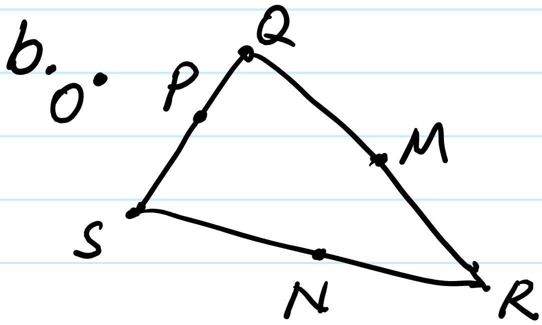
$\sqrt{h}=\frac{4}{5}$

$h=\frac{16}{25}$ $\boxed{1\text{ mark}}$

$\boxed{1\text{ mark}}$

the height from the top is $\frac{\sqrt{3}}{2}-\frac{16}{25}=0.23\,m.$

b.



$\overrightarrow{OQ}=q$, $\overrightarrow{OS}=\underset{\sim}{s}$

$\overrightarrow{QP}=\frac{1}{2}(\overrightarrow{QS})$

$=\frac{1}{2}(\overrightarrow{OS}-\overrightarrow{OQ})$

$=\frac{1}{2}(\underset{\sim}{s}-\underset{\sim}{q})$ $\boxed{1\text{ mark}}$

$\overrightarrow{MN}=\overrightarrow{MR}+\overrightarrow{RN}$

$=\frac{1}{2}\overrightarrow{QR}+\frac{1}{2}\overrightarrow{RS}$

$=\frac{1}{2}(\overrightarrow{QR}+\overrightarrow{RS})$ $\boxed{1\text{ mark}}$

$=\frac{1}{2}(\overrightarrow{OR}-\overrightarrow{OQ}+\overrightarrow{OS}-\overrightarrow{OR})$

$=\frac{1}{2}(\overrightarrow{OS}-\overrightarrow{OQ})=\frac{1}{2}(\underset{\sim}{s}-\underset{\sim}{q})$

since $\overrightarrow{QP}=\overrightarrow{MN}$ $\boxed{1\text{ mark}}$

$\therefore$ QMNP is a parallelogram.

c) $X \sim B(15, 0.08)$

i) $P(X=2) = {}^{15}C_2 \times 0.08^2 \times 0.92^{13}$

$= 0.227$ $\boxed{1 \text{ mark}}$

ii) production stops if $X > 2$

$P(X>2) = 1 - P(X \leq 2)$

$\boxed{1 \text{ mark}}$ $= 1 - (P(X=0) + P(X=1) + P(X=2))$

$= 1 - (0.92^{15} + {}^{15}C_1 0.08 \times 0.92^{14} + 0.227306)$

$P(X>2) = 0.113$ $\boxed{1 \text{ mark}}$